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On the Infinitely Long Cylindrical Antenna*

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Using the method of steepest descents the far field and the asymptotic form of the current distribution is obtained for an infinitely long, perfectly conducting cylindrical antenna excited by a localized electromotive force. The low frequency value of the radiation conductance is determined by integrating the radiated energy flux over a large sphere.

I. INTRODUCTION

THE problem of determining the longitudinal distribution of current and the input impedance of an infinitely long, perfectly conducting circular cylinder antenna has interested several investigators. Among them are Stratton and Chu,[1] Schelkunoff,[2] and Hallén.[3] And recently a new method of analyzing the problem was brought to the author's attention through an unpublished note by H. Levine in which he elegantly formulates the input impedance by means of the free-space dyadic Green's function and a variational principle involving the current. Using a simple plane wave-trial function, he obtains an expression for the input impedance which is valid for a broader frequency range than the expressions of the aforementioned investigators.

Unless one uses the variational approach, the core of the problem lies in the evaluation of a certain contour integral. Once this integral has been evaluated the current at any point along the antenna and hence its input impedance is readily determined. In this paper, by means of the method of steepest descents we evaluate this integral for distances far from the localized electromotive force exciting the antenna. This gives the explicit form of the far-zone field from which the asymptotic value of the antenna current is easily calculated. Moreover, the input conductance of the antenna is obtained by integrating the radiated energy flux over a large sphere.

To formulate the problem we assume that an infinitely long, perfectly conducting circular cylinder of radius a is driven at some cross section by a localized electromotive force, V, circumscribing the cylinder in a narrow peripheral band. Since only longitudinal currents can be generated by such a source, the axial and radial components of the magnetic vector H and the circumferential component of the electric vector E are identically zero. That is,

$$E=(E_\rho,\, 0,\, E_z) \tag{1}$$

$$H=(0,\, H_\phi,\, 0) \tag{2}$$

where E_ρ, H_ϕ, E_z are the non-vanishing components of the field in cylindrical coordinates (Fig. 1). The exact expression for $E_z(\rho, \phi, z)$ is given by

$$E_z(\rho,\phi,z)=\frac{V}{2\pi}\int_{-\infty}^{\infty}\frac{H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}\rho]}{H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}a]}e^{i\zeta z}d\zeta \tag{3}$$

where $H_0^{(1)}(x)$ is the Hankel function of the first kind and zero order, $k=2\pi/\lambda$, λ being the free-space wave-length, and ζ is a complex variable. The path of integration is along the real axis of the complex ζ-plane with an upward indentation at $\zeta=-k$ and a downward one at $\zeta=+k$ (Fig. 2). This equation can be obtained from previous investigations,[1,2,3] or from a report by Papas[4] in which he derives integral expressions for the field components and the antenna current by means of a Green's function used by Levine and Schwinger.[5] Our first task is to find the explicit form of Eq. (3) for large values of r.

II. FAR FIELD

To find the far field, i.e., find E_z when r is large, the method of steepest descents[6] is used. That is, the integral

$$E_z(\rho,\phi,z)=\frac{V}{2\pi}\int_{-\infty}^{\infty}\frac{H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}\rho]}{H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}a]}e^{i\zeta z}d\zeta \tag{3}$$

is evaluated for large r. Let $u=(k^2-\zeta^2)^{\frac{1}{2}}$; then for ρ large and $u\neq 0$,

$$H_0^{(1)}(u\rho)e^{i\zeta z}\sim\left\{\frac{2}{\pi u\rho}\right\}^{\frac{1}{2}}\exp\left[i\left(u\rho-\frac{\pi}{4}+\zeta z\right)\right]. \tag{4}$$

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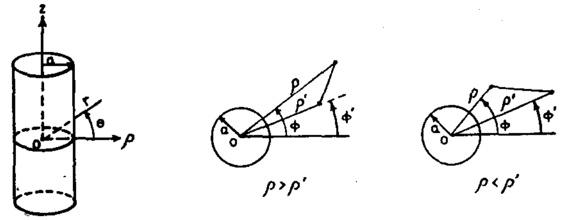

FIG. 1. Coordinate systems used in the formulation of the problem. Cylindrical coordinates: ρ, ϕ, z. Spherical coordinates: r, θ, ϕ.

The value of ζ which makes the exponential $e^{i(u\rho+\zeta z)}$ an extremum, is a saddlepoint and is denoted by ζ_0. Hence ζ_0 is the solution of

$$\frac{d}{d\zeta}[(k^2-\zeta^2)^{\frac{1}{2}}\rho+\zeta z]=0.$$

Carrying out the differentiation, it is easily found that

$$\zeta_0=\frac{kz}{(\rho^2+z^2)^{\frac{1}{2}}}=k\sin\theta \tag{5}$$

where θ is measured from the plane $z=0$. At $\zeta=\zeta_0$

$$e^{i(u\rho+\zeta z)}=e^{ik(\rho\cos\theta+z\sin\theta)}$$

and in the neighborhood of ζ_0 it is approximately equal to

$$\exp\left[ik(\rho\cos\theta+z\sin\theta)-\frac{i\rho}{2k\cos^3\theta}(\zeta-\zeta_0)^2\right]. \tag{6}$$

The path of integration through the saddlepoint is determined by the constancy of the imaginary part of the exponent of Eq. (6); that is, the path is so chosen that the imaginary part of the exponent remains constant as the path of integration traverses the saddlepoint. Since the exponent is a monogenic function and its imaginary part remains constant, its real part rises from negligibly small values to a peak value at the saddlepoint and then down again. The requirement for the constancy of the imaginary part of the exponent is expressed by

$$\text{I.P.}[ik(\rho\cos\theta+z\sin\theta)]$$
$$=\text{I.P.}\left[ik(\rho\cos\theta+z\sin\theta)-\frac{i\rho(\zeta-\zeta_0)^2}{2k\cos^3\theta}\right]. \tag{7}$$

Let $\zeta-\zeta_0=se^{i\alpha}$. In this, α is the angle which the path makes with the real axis of the ζ-plane. Substituting this into Eq. (7), it is found that if (7) is to be satisfied, it is necessary that the R.P. $(s^2e^{2i\alpha})=0$; i.e., $\cos2\alpha=0$ or $\alpha=\pm\pi/4$. The value $\alpha=\pi/4$ is excluded because the path through ζ_0 must extend from the second to the fourth quadrant of the complex ζ-plane; this leaves $\alpha=-\pi/4$ as the angle which the path of steepest descent makes with the real axis of the complex ζ-plane.

We now transform[7] from the ζ-plane to the complex τ-plane by the following transformation

$$\zeta=k\sin\tau$$
$$\tau=\theta+i\psi. \tag{8}$$

The path of integration in the ζ-plane (Fig. 2) transforms into path C in the τ-plane of Fig. 3. The saddlepoints lie on the real axis of the τ-plane, i.e., $\tau=\theta$ is a saddlepoint. We de-form C into C_0, which starts at $-(\pi/2)+i\infty$, goes through $\theta=0$, and then ends at $\pi/2+i\infty$. A path such as C_θ passing through a saddlepoint θ would have to start at $\theta-(\pi/2)+i\infty$ and terminate at $\theta+(\pi/2)+i\infty$. In the τ-plane, Eq. (4) transforms into

$$\left(\frac{2}{\pi\rho k\cos\tau}\right)^{\frac{1}{2}}e^{-i(\pi/4)}e^{ik(\rho\cos\tau+z\sin\tau)}$$
$$=\left(\frac{2}{\pi\rho k\cos\tau}\right)^{\frac{1}{2}}e^{-i(\pi/4)}e^{ikr\cos(\tau-\theta)}. \tag{9}$$

Let $F(k\cos\tau)$ represent the factor

$$\{H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}a]\}^{-1}$$

in the τ-plane. Substituting this and Eq. (9) into (3), we get

$$E_z(\rho,z)=\frac{V}{\pi}e^{-i(\pi/4)}\int_{\theta-(\pi/2)+i\infty}^{\theta+(\pi/2)-i\infty}\left\{\frac{2}{\pi\rho k\cos\tau}\right\}^{\frac{1}{2}}$$
$$\times e^{ikr\cos(\tau-\theta)}F(k\cos\tau)k\cos\tau d\tau. \tag{10}$$

This integration is carried out along the path C_θ. It is only necessary to carry out the integration on a short segment of the path in the neighborhood of $\tau=\theta$. We set $\tau-\theta=\eta e^{-i(\pi/4)}$ where η is the distance along the path measured from the saddlepoint. When $|\tau-\theta|$ is small,

$$\cos(\tau-\theta)\doteq1-\frac{(\tau-\theta)^2}{2}=1-\frac{\eta^2}{2}e^{-i(\pi/2)}.$$

Moreover, $d\tau=d\eta e^{-i(\pi/4)}$. With these, Eq. (10) takes the form

$$E_z(\rho,z)=-\frac{Vi}{\pi}e^{ikr}F(k\cos\theta)$$
$$\times\left\{\frac{2k\cos\theta}{\pi\rho}\right\}^{\frac{1}{2}}\int_{-\epsilon}^{\epsilon}e^{-(kr\eta^2/2)}d\eta. \tag{11}$$

Since r is large we can replace the limits, $-\epsilon$ to ϵ,

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[7] F. Noether, "Elektromagnetische Wellen an einem Draht, bei konzentrierter Energiequelle;" Physik Zeits. USSR, Band 8, Heft 1, 1–24 (1935).

by infinite ones. That is,

$$\int_{-\epsilon}^{\epsilon} e^{-(kr\eta^2/2)}d\eta \doteq \int_{-\infty}^{\infty} e^{-(kr\eta^2/2)}d\eta = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}}. \quad (12)$$

Recalling that $r\cos\theta=\rho$, Eq. (11) takes the form

$$E_z(r,\theta) = -\frac{Vi}{\pi}\frac{e^{ikr}}{r}\frac{1}{H_0^{(1)}(ka\cos\theta)}. \quad (13)$$

This is the far field of an infinitely long antenna driven by a localized e.m.f. V. The θ-component of the E-field and the ϕ-component of the H-field are given respectively by

$$E_\theta(r,\theta) = \frac{E_z(r,\theta)}{\cos\theta} = -\frac{Vi}{\pi}\frac{e^{ikr}}{r\cos\theta}\frac{1}{H_0^{(1)}(ka\cos\theta)} \quad (14)$$

and

$$H_\phi(r,\theta) = \frac{E_\theta(r,\theta)}{120\pi} = -\frac{Vi}{120\pi^2}\frac{e^{ikr}}{r\cos\theta} \times \frac{1}{H_0^{(1)}(ka\cos\theta)}. \quad (15)$$

III. THE RADIATION CONDUCTANCE

To find the radiation conductance of the antenna, the radial component of Poynting's vector S must be integrated over the surface of a large sphere. This integration yields the radiated power P, and division by $V^2/2$ gives the radiation conductance.

Poynting's vector is given by

$$S = \frac{1}{2}E_\theta H_\phi^* = \frac{V^2}{240\pi^3}\frac{1}{r^2\cos^2\theta} \times \frac{1}{[H_0^{(1)}(ka\cos\theta)][H_0^{(1)}(ka\cos\theta)]^*} \quad (16)$$

where the asterisk indicates conjugate complex values. Thus

$$P = \int_{-\pi/2}^{\pi/2} 2\pi r^2 S\cos\theta d\theta = \frac{V^2}{120\pi^2}\int_{-\pi/2}^{\pi/2}\frac{1}{\cos\theta} \times \frac{d\theta}{[H_0^{(1)}(ka\cos\theta)][H_0^{(1)}(ka\cos\theta)]^*} \quad (17)$$

when ka is small,

$$H_0^{(1)}(ka\cos\theta) \rightarrow 1 - \frac{2i}{\pi}\log\frac{2}{\gamma ka\cos\theta},$$

and the integral in Eq. (17) is approximately

FIG. 2. Paths of integration in the ζ-plane.

equal to

$$2\int_0^{\pi/2}\frac{d\theta}{\cos\theta\left|1-\frac{2i}{\pi}\log\frac{2}{\gamma ka\cos\theta}\right|^2} = 2\int_0^{\pi/2} \times \frac{d\theta}{\cos\theta\left|1-\frac{2i}{\pi}\left(\log\frac{2}{\gamma}-\log ka-\log\cos\theta\right)\right|^2} \quad (18)$$

where $\log 2/\gamma = 0.1159$. This integral appears to be rather difficult to handle rigorously, and we therefore resort to an approximation.

Let us examine the integrand

$$\frac{1}{\cos\theta\left|1-\frac{2i}{\pi}\left(\log\frac{2}{\gamma}-\log ka-\log\cos\theta\right)\right|^2}. \quad (19)$$

Since ka is small, the term in square brackets can be replaced by $(4/\pi^2)(\log ka+\log\cos\theta)^2$. This quantity varies from $(4/\pi^2)(\log ka)^2$ to $(16/\pi^2)(\log ka)^2$ as θ varies from $\theta=0$ to $\theta=\theta_1=\text{arc}\cos ka$. For this range we replace $(4/\pi^2)(\log ka+\log\cos\theta)^2$ by the geometric mean of its end values, i.e. $(8/\pi^2)(\log ka)^2$. Hence for the range $\theta\leq\theta\leq\theta_1$, the contribution to Eq. (18) is

$$2\int_0^{\theta_1}\frac{\pi^2}{8}\frac{1}{[\log ka]^2}\frac{d\theta}{\cos\theta} = \frac{\pi^2}{4}\frac{1}{[\log ka]^2} \times \log\tan\left(\frac{\pi}{4}+\frac{\theta_1}{2}\right) \doteq \frac{\pi^2\log\operatorname{ctn}\left(\frac{ka}{2}\right)}{4[\log ka]^2}, \quad (20)$$

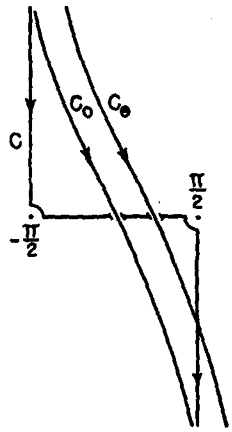

FIG. 3. Paths of integration in the τ-plane.

since $\theta_1=\arccos ka \doteq \pi/2-ka$. Moreover $\log \operatorname{ctn} ka/2 \doteq \log 2/ka = \log 2 + \log 1/ka$. Neglecting the term $\log 2$, (20) is approximately equal to

$$2\int_0^{\theta_1} \frac{\pi^2}{8} \frac{1}{[\log ka]^2} \frac{d\theta}{\cos\theta} \doteq \frac{\pi^2}{4} \frac{1}{\log\frac{1}{ka}}. \tag{21}$$

To complete the range of integration we again approximate the integrand Eq. (19)

$$2\int_{\theta_1}^{\pi/2} \frac{d\theta}{\cos\theta\left[\frac{2}{\pi}\log\frac{2}{\gamma ka\cos\theta}\right]^2} \doteq \frac{\pi^2}{2}\int_{\theta_1}^{\pi/2}$$

$$\times \frac{d\theta}{\left(\frac{\pi}{2}-\theta\right)\left(\log\left\{ka\left(\frac{\pi}{2}-\theta\right)\right\}\right)^2}. \tag{22}$$

This integral can be immediately evaluated and has the value $(\pi^2/4)[1/\log(1/ka)]$. By adding this result to Eq. (21) the approximate value of (17) is obtained. Hence,

$$P=\frac{V^2}{120\pi^2}\left[\frac{\pi^2}{4}\frac{1}{\log\frac{1}{ka}}+\frac{\pi^2}{4}\frac{1}{\log\frac{1}{ka}}\right].$$

The radiation conductance G is given by

$$G=\frac{2P}{V^2}=\frac{1}{120\log\left(\frac{1}{ka}\right)}. \tag{23}$$

On physical grounds G must be positive. Consequently it is necessary that $1/ka>1$ or $\lambda>2\pi a$. ($k=2\pi/\lambda$ where λ is the wave-length of the exciting source). This means that Eq. (23) is a low frequency, i.e., long wave-length, approximation to the radiation conductance. When $ka=1$, G becomes infinite.

IV. ASYMPTOTIC FORM OF THE CURRENT

For great distances from the localized e.m.f. the current distribution can be computed from the far-field expression (15). For large values of r we found that

$$H_\phi(r,\theta)=-\frac{Vi}{120\pi^2}\frac{e^{ikr}}{r\cos\theta}\frac{1}{H_0^{(1)}(ka\cos\theta)}. \tag{24}$$

To obtain the value of H_ϕ on the surface of the cylinder for large z, it is necessary to restrict r and θ so that $r\cos\theta=a$; moreover, $r\sin\theta=z$. But since $\theta\doteq\pi/2$, we have $r\doteq z$. The argument of the Hankel function is given by the approximation: $ka\cos\theta \doteq ka^2/z$. Substituting these into Eq. (24) we obtain H_ϕ for $\rho=a$ and z large:

$$\underset{\substack{\rho=a\\ z\to\infty}}{(H_\phi)} = -\frac{Vi}{120\pi^2}\frac{e^{ikz}}{a}\frac{1}{H_0^{(1)}\left(\frac{ka^2}{z}\right)}. \tag{25}$$

The current for large z is then given by

$$I(z)=2\pi aH_\phi=-\frac{Vi}{60\pi}\frac{e^{ikz}}{H_0^{(1)}\left(\frac{ka^2}{z}\right)}. \tag{26}$$

Since

$$\frac{ka^2}{z}\ll 1,\quad H_0^{(1)}\left(\frac{ka^2}{z}\right)\doteq -\frac{2i}{\pi}\log\frac{2z}{\gamma ka^2}.$$

Using this approximation the asymptotic form of the current takes the form

$$I(z)=\frac{V}{120}\frac{e^{ikz}}{\log\frac{z}{ka^2}}. \tag{27}$$

V. CONCLUSIONS

The far field (Eqs. (14), (15)) and the asymptotic form of the current at large distances from the source of excitation (Eq. (27)) were determined by the method of steepest descent. By integrating the radiated energy flux an approximate value of the radiation conductance was calculated (Eq. (23)). This integration, however, yields no information about the radiation susceptance since the sphere over which the integration is performed lies in the far zone where the time average energy flux is purely real. There is close agreement for long wave-lengths, i.e., for low frequencies, among the values of conductance obtained by Schelkunoff,[2] Hallén,[3] and Eq. (23). For shorter wave-lengths, Eq. (23) becomes invalid since the radiation conductance becomes infinite when $\lambda=2\pi a$ and negative when $\lambda<2\pi a$.

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Note added in proof: Professor S. Silver has also derived (3). See Report No. 149, Antenna Lab., University of California, Jan. 3, 1949, "The Field of a Slot of Arbitrary Shape In an Infinite Cylinder."